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Two-dimensional filling in ordered and disordered systems

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Abstract. We consider filling or wedge-wetting transitions occurring in a (1 + 1)-dimensional wedge geometry with both thermal and random-bond disorder using effective interfacial Hamiltonian models. For ordered systems the problem may be solved using transfer-matrix methods for quite arbitrary choices of interfacial binding potential and gives a complete classification of the possible critical behaviours. For random bonds the transition is studied for short-ranged forces using the replica trick and the wedge-wetting critical exponents are identified. Our results establish a remarkable relation between the mid-point height probability distribution $P_F(l_0)$ at filling transitions and the appropriately defined height distribution function $P_\pi(l;\theta_\pi)$ at planar wetting transitions. We observe that provided the wetting specific heat component $\alpha_s = 0$, then, in the scaling limit, $P_F(l_0) = P_\pi(l;\theta_\pi - \alpha)$ where θ_π is the contact angle and α is the tilt angle of the wedge. This relation completely determines the allowed values of the filling critical exponents in the fluctuation-dominated regimes. Conjectures regarding interfacial fluctuation effects in finite-size two-dimensional systems are also made.

1. Introduction

In this article we investigate the nature of fluctuation effects occurring at two-dimensional filling or wedge-wetting transitions in the presence of bulk disorder (random-bond impurities). Our purpose is to extend recent studies of fluid adsorption in two- and three-dimensional wedges with purely thermal excitations, which have highlighted the remarkably strong influence of interfacial fluctuations on the critical singularities at filling transitions [1, 2]. In general, the fluctuation-related properties of filling transitions are quite different to those predicted to occur for wetting at a planar wall. However, in two dimensions we shall show from explicit transfer-matrix-based calculations that for systems with short-ranged forces (belonging to the most strongly fluctuation-dominated regimes) and in the presence of marginal long-ranged interactions there is an equivalence of the (appropriately defined) critical exponents for filling and wetting. This equivalence extends to the detailed scaling properties for the interfacial height distribution function and is valid for ordered and disordered systems. This is strongly suggestive that properties of interfacial fluctuations occurring in different two-dimensional geometries satisfy a covariance relation. To continue our introduction we present some further details concerning the phenomenology and critical singularities characteristic of filling transitions and provide a brief synopsis of our paper.

Consider a wedge (in $d = 3$ dimensions) formed by the union of two flat walls tilted at angles $\pm\alpha$ to the horizontal. Axes (x, y) are oriented across and along the wedge respectively and the whole system is supposed to be in contact with a bulk vapour phase at temperature T and chemical potential μ tuned so that the system is at bulk liquid–vapour coexistence

($\mu = \mu_{\rm sat}$). Macroscopic thermodynamic arguments [3–5] show that the wedge is completely filled with liquid for temperatures $T \geqslant T_F$, where the filling temperature T_F is specified by

$$\theta_\pi(T_F) = \alpha \tag{1.1}$$

and $\theta_\pi(T)$ denotes the contact angle of a liquid drop on a planar substrate. Consequently the filling temperature T_F is necessarily lower than any wetting transition temperature T_W characteristic of the planar wall–vapour interface. Indeed filling may happen in the absence of wetting transition, i.e. even if partial wetting is present up to the bulk critical temperature T_c. The filling transition occurring as $T \to T_F^-$ may be first or second order, corresponding to the discontinuous or continuous divergence of the mid-point interfacial height l_0 (measured from the bottom of the wedge). Importantly, the order of the filling transition is not necessarily the same as that of the wetting transition present in the planar system. In particular, continuous filling transitions can occur in rather general circumstances even if the walls of the wedge exhibit first-order wetting [2]. This means that the observation of continuous filling transitions is a realistic experimental possibility even in the absence of any known solid–fluid interfaces exhibiting continuous (critical) wetting.

From a theoretical perspective, continuous filling transitions are also highly interesting because the manifestations of the soft-mode interfacial-like fluctuations are much stronger (in general) than that predicted to occur at continuous wetting [6–9]. The reasons for this can be traced to the extreme anisotropy of these fluctuations in the wedge geometry which lower the effective dimensionality of the soft interfacial mode [2]. Consider for example the interfacial height–height correlation function which is characterized by correlation lengths across (ξ_x) and along (ξ_y) the wedge. Effective interfacial Hamiltonian models [2] of filling predict that as $T \to T_F^-$ the correlation length ξ_y diverges much faster than ξ_x (which is always comparable with the height l_0). This implies that the dominant fluctuations at three-dimensional filling have a one-dimensional character with a soft-mode corresponding to long-wavelength fluctuations of the local filling height l_0 in the y-direction. On the basis of this picture a general fluctuation theory for three-dimensional filling has been proposed [2] and leads to the following predictions for the critical exponents describing the divergence of the interfacial height $l_0 \sim (T_F - T)^{-\beta_0}$, roughness $\xi_\perp \sim (T_F - T)^{-\nu_\perp}$ and correlations length $\xi_y \sim (T_F - T)^{-\nu_y}$. There are two fluctuation regimes depending on the exponent describing the range of the intermolecular forces p (see below). In the mean-field regime ($p < 4$)

$$\beta_0 = \frac{1}{p} \qquad \nu_y = \frac{1}{2} + \frac{1}{p} \qquad \nu_\perp = \frac{1}{4} \tag{1.2}$$

whilst in the fluctuation-dominated regime ($p > 4$)

$$\beta_0 = \frac{1}{4} \qquad \nu_y = \frac{3}{4} \qquad \nu_\perp = \frac{1}{4}. \tag{1.3}$$

These critical exponents are totally different those pertaining to wetting in three dimensions [6, 8] and in particular predict a universal roughness exponent, $\nu_\perp = \frac{1}{4}$, independent of the range of the forces.

The general fluctuation theory presented in [2] can be easily generalized to other dimensions and in particular for the two-dimensional wedge leads to the predictions

$$\beta_0 = 1 \qquad \nu_\perp = 1 \qquad \text{for } p > 1 \tag{1.4}$$

and

$$\beta_0 = \frac{1}{p} \qquad \nu_\perp = \frac{1+p}{2p} \qquad \text{for } p < 1 \tag{1.5}$$

which again fall into fluctuation-dominated and mean-field regimes respectively. This behaviour is fully consistent with the results of detailed transfer-matrix calculations of an inhomogeneous effective Hamiltonian model of two-dimensional ($d = 2$) filling [1].

The values of the filling transition critical exponents for $d = 2$ are certainly not identical to those pertaining to critical wetting in the same dimensions. Recall that the critical singularities of the wetting transition fall into one of three possible fluctuation classes (labelled the strong-, weak- and mean-field-fluctuation regimes) with the universality only characteristic of the former corresponding to $p > 2$ [7, 9, 10]. Nevertheless within the strong-fluctuation regime for two-dimensional critical wetting the critical exponents describing the growth of the planar liquid film thickness $l^{(\pi)} \sim (T_w - T)^{-\beta_s}$ and roughness $\xi_\perp^{(\pi)} \sim (T_w - T)^{-\nu_\perp^{(\pi)}}$ have the same numerical values (unity) [10, 11] as the critical exponents β_0 and $\nu_\perp$ pertinent to a two-dimensional fluctuation-dominated filling transition [1]. Here we denote the wetting exponent as $\nu_\perp^{(\pi)}$ so as to distinguish it from the exponent $\nu_\perp$ defined for filling. The question is, is this a coincidence or is there a deeper connection between filling and wetting for $d = 2$ for systems with short-ranged forces?

To test this latter possibility we have extended Kardar's Bethe-*ansatz* study [7, 12] of two-dimensional wetting with random bonds to filling in a wedge geometry. Fortunately, the method generalizes quite easily and we are able to establish the existence of a filling transition at temperature $T_F < T_w$ which is exactly in accordance with the thermodynamic prediction. Moreover, we show that the mid-point height distribution $P_F(l_0)$ for filling has an identical scaling structure to that found for wetting with random bonds for $d = 2$, so the critical exponents β_0 and $\nu_\perp$ are once again identical to those for wetting. This invariance of the probability distribution is elegantly expressed by the relation

$$P_F(l_0) = P_\pi(l; \theta_\pi - \alpha) \tag{1.6}$$

where $P_F(l_0)$ denotes the equivalent probability distribution for the case of critical wetting at a planar substrate wall with contact angle θ_π. This invariance is valid for both ordered and disordered systems and is indicative of a fundamental relationship between the manifestation of interfacial fluctuations in the two-dimensional wedge and planar substrate geometries respectively. We believe this observation to be non-trivial and, if more generally valid, to have important implications for other systems.

Our article is arranged as follows. In section 2 we review the theory of filling in two-dimensional systems without bulk disorder where the identification (1.6) can be readily established. Some of these results have been reported briefly before [1] but a fuller presentation is given here. In section 3 we turn to the main technical part of our analysis and show how Kardar's Bethe-*ansatz* approach works equally well for the filling transition and naturally leads to the result for the probability distribution quoted above. From this we observe the importance of the value of the wetting specific heat exponent $\alpha_s = 0$ and then return to the ordered bulk problem to discuss the case of marginal long-ranged forces. We conclude by discussing the implications of our work and make a conjecture based on a generalization of (1.6) concerning the structure of the probability distribution for finite-size effects at filling transitions. These should be verifiable in future numerical transfer-matrix and/or simulation studies.

2. Wetting and filling in ordered systems (I)—general forces

To begin, it is worth recalling a few results connected with wetting at planar walls before we proceed to the case of filling. The standard fluctuation theory of the transition in 1 + 1 dimensions is based on the effective Hamiltonian model [6–11]

$$H[l] = \int \mathrm{d}x \left\{ \frac{\Sigma}{2}\left(\frac{\partial l}{\partial x}\right)^2 + W(l) \right\} \tag{2.1}$$

where $l(x)$ is the local interfacial height at position x along the wall, $W(l)$ is the binding potential and Σ denotes the interfacial stiffness of the unbinding interface. We will concentrate on fluid interfaces for which Σ may be identified with the liquid–vapour surface tension. At bulk liquid–vapour coexistence ($\mu = \mu_{\mathrm{sat}}$), the binding potential is taken to have the general algebraic form

$$W(l) = -\frac{a}{l^p} + \frac{b}{l^q} \qquad l > 0 \tag{2.2}$$

with a, b effective Hamaker constants and exponents $q > p$ which denote the range of the forces. It is well known [11] that the partition function $Z_\pi(l_1, l_2; X)$ for an interface of length X with end positions $l(0) = l_1$ and $l(x) = l_2$ can be expressed as a spectral sum (or integral if scattering states are present)

$$Z_\pi(l_1, l_2; X) = \sum_{n=0}^{\infty} \psi_n(l_1)\psi_n(l_2)\mathrm{e}^{-E_n X} \tag{2.3}$$

where the eigenfunctions and eigenvalues satisfy the Schrödinger equation

$$-\frac{1}{2\Sigma}\frac{\partial^2 \psi_n(l)}{\partial l^2} + W(l)\psi_n(l) = E_n \psi_n(l) \tag{2.4}$$

and we have set $k_B T = 1$ for convenience. Thus, in the thermodynamic limit $X \to \infty$, the excess free energy of the wall–vapour interface $f_{\mathrm{sing}} = \sigma_{wv} - (\sigma_{wl} + \Sigma)$, defined in terms of the wall–vapour and wall–liquid surface tensions is simply

$$f_{\mathrm{sing}} = E_0 \tag{2.5}$$

from which we can identify the contact angle

$$\theta_\pi = \sqrt{\frac{-2E_0}{\Sigma}} \tag{2.6}$$

via Young's equation $f_{\mathrm{sing}} = \Sigma(\cos\theta_\pi - 1)$ in the small-contact-angle limit, for which the model (2.1) is valid. Recall that at a wetting transition the singular free energy vanishes as $f_{\mathrm{sing}} \sim (T_w - T)^{2-\alpha_s}$, so $\theta_\pi \sim (T_w - T)^{(2-\alpha_s)/2}$. Similarly the normalized probability distribution for the interfacial height $P_\pi(l)$ is

$$P_\pi(l) \equiv |\psi_0(l)|^2 \tag{2.7}$$

corresponding to the standard quantum mechanical result. For later purposes it is also convenient to define the matrix elements

$$\langle m|f(l)|n\rangle \equiv \int \mathrm{d}l\ \psi_m^*(l) f(l) \psi_n(l) \tag{2.8}$$

which will appear in our description of the wedge geometry. Finally we note that the transverse correlation length is given by

$$\xi_\parallel \sim (E_1 - E_0)^{-1} \tag{2.9}$$

and characterizes correlations in the fluctuations of the interfacial height along the wall. As remarked in the introduction, the fluctuation theory of wetting generally predicts the existence of three fluctuation regimes for fixed dimensionality $d < 3$ [10]. In particular, for $d = 2$ the critical behaviour belongs to the:

(i) Strong-fluctuation (SFL) regime if $p > 2$. The critical exponents are universal with $\alpha_s = 0$, $\beta_s = \nu_\perp^{(\pi)} = 1$ and $\nu_\parallel = 2$.
(ii) Weak-fluctuation (WFL) regime if $p < 2$ but $q > 2$. The critical exponents are non-universal with $\alpha_s = 2(1 - p)/(2 - p)$, $\beta_s = \nu_\perp^{(\pi)} = 1/(2 - p)$ and $\nu_\parallel = 2/(2 - p)$.
(iii) Mean-field (MF) regime if $q < 2$. The critical exponents are again non-universal with $\alpha_s = (q - 2p)/(q - p)$, $\beta_s = 1/(q - p)$ and $2\nu_\perp^{(\pi)} = \nu_\parallel = (q + 2)/[2(q - p)]$. Note that for this regime $\beta_s > \nu_\parallel$, so the transition is not fluctuation dominated.

Two other marginal or intermediate-fluctuation regions are also possible corresponding to the boundaries of the regimes outlined above. These correspond to the cases in which either the leading-order or next-to-leading-order decay terms are of order l^{-2}. At the SFL/WFL boundary this has a dramatic influence on the behaviour, leading to three more sub-regimes, but at the WFL/MF boundary this corresponds to the potential

$$W(l) = -\frac{a}{l^p} + \frac{w}{l^2}. \tag{2.10}$$

The critical exponents α_s, β_s, $\nu_\perp^{(\pi)}$ and $\nu_\parallel$ are unchanged from those quoted above for the WFL regime. We will return to this case later in this article.

Before we turn to the case of filling we comment on the structure of the interface height probability distribution function in the SFL regime. The scaling behaviour of $P_\pi(l)$ characteristic of this regime emerges directly if we drop the potential $W(l)$ in (2.4) and use the boundary condition [11]

$$\left.\frac{\partial \ln \psi_0}{\partial l}\right|_{l=0} = -\lambda \tag{2.11}$$

where $\lambda \propto (T_w - T)$ measures the linear distance from the wetting temperature. Thus we obtain

$$P_\pi(l) = \frac{1}{l_\pi} \mathrm{e}^{-l/l_\pi} \qquad \text{SFL} \tag{2.12}$$

where $l_\pi \equiv \langle l \rangle \sim (T_w - T)^{-1}$ is the average interfacial height. This expression is valid in the scaling limit $T \to T_w^-$, $l \to \infty$ with l/l_π arbitrary. For later purposes it is convenient to rewrite this result in terms of the contact angle $\theta_\pi \sim (T_w - T)$ to make its dependence explicit. Thus we write

$$P_\pi(l; \theta_\pi) = 2\Sigma\theta_\pi \mathrm{e}^{-2\Sigma\theta_\pi l} \qquad \text{SFL} \tag{2.13}$$

which is precisely equivalent to (2.12).

We now turn to the wedge geometry and outline the transfer-matrix calculation presented in [1]. To extract the quantities of interest some care is needed with the thermodynamic limit and we first consider a periodic wedge geometry defined over the horizontal range $[-X/2, X/2]$; see figure 1. The local height of the interface from the horizontal axis is written as $y(x)$ with periodic boundary conditions, $y(X/2) = y(-X/2)$, and the height of the wall itself is written as $z(x) \equiv \alpha|x|$. The relative local separation of the two is denoted as $l(x) \equiv y(x) - z(x)$. For open wedges corresponding to small α (such that $\tan \alpha \approx \alpha$), it is already known from previous mean-field analysis [5] that an appropriate effective Hamiltonian is

$$H[y] = \int \mathrm{d}x \, \left\{ \frac{\Sigma}{2} \left(\frac{\mathrm{d}y}{\mathrm{d}x} \right)^2 + W(y - z) \right\}. \tag{2.14}$$

The partition function for this periodic wedge system is the functional integral over all configurations

$$Z_p = \int \mathcal{D}y \, \mathrm{e}^{-H[y]} \tag{2.15}$$

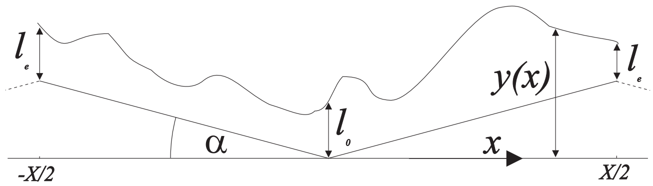

Figure 1. A schematic illustration of the wedge geometry in 1 + 1 dimensions with periodic boundary conditions. The notation is defined in the text.

from which we calculate the free energy

$$F_p = -\ln Z_p. \tag{2.16}$$

The advantage of this choice of boundary conditions is that it allows us to easily extract the excess wedge free energy $F_w(\alpha)$. To see this, note that in the thermodynamic limit $X \to \infty$, the periodic system reduces to two independent wedges with angles α and $-\alpha$ respectively. The excess free energy is conveniently defined by

$$f_w(\alpha) + f_w(-\alpha) = \lim_{X\to\infty} (F_p - F'_p) = -\lim_{X\to\infty} \ln\left(\frac{Z_p}{Z'_p}\right) \tag{2.17}$$

where $F'_p \equiv -\ln(Z'_p)$ denotes the free energy of a single wall of horizontally projected length X, tilted at an angle α to the horizontal (with shifted periodic boundary conditions $y(X/2) \equiv y(-X/2) + \alpha X$). Equation (2.17) shows the independent contributions from the wedge and inverted wedge arising in the thermodynamic limit. The partition function Z_p is equivalent to a fluctuation sum over all graphs $l(x) = y(x) - z(x)$. Making this change of variable, we rewrite the Hamiltonian as

$$\tilde{H}[l] = \frac{\Sigma\alpha^2 X}{2} + 2\Sigma\alpha(l_0 - l_e) + \int_{-X/2}^{X/2} \mathrm{d}x \left\{\frac{\Sigma}{2}\left(\frac{\partial l}{\partial x}\right)^2 + W(l)\right\} \tag{2.18}$$

where $l_0 \equiv y(0)$ is the mid-point height and $l_e \equiv y(X/2) - z(X/2)$ is the edge (relative) height. Consequently

$$Z_p = \mathrm{e}^{\Sigma\alpha^2 X/2} \int\int \mathrm{d}l_0\, \mathrm{d}l_e\, Z_\pi\left(l_e, l_0, \frac{X}{2}\right) \mathrm{e}^{2\Sigma\alpha(l_0 - l_e)} Z_\pi\left(l_0, l_e, \frac{X}{2}\right) \tag{2.19}$$

whilst

$$Z'_p = \mathrm{e}^{\Sigma\alpha^2 X/2} \int \mathrm{d}l_e\, Z_\pi(l_e, l_e, X). \tag{2.20}$$

Substituting for the quantum mechanical spectral sum, we find in the thermodynamic limit that

$$f_w(\alpha) = -\ln\langle 0|\mathrm{e}^{2\Sigma\alpha l_0}|0\rangle \tag{2.21}$$

where the inner product is defined in terms of the planar system eigenfunctions as in equation (2.8). From the wedge free energy it is straightforward to calculate the mean value of the mid-point height from (2.19):

$$\langle l_0\rangle = -\frac{1}{2\Sigma}\frac{\partial}{\partial\alpha} f_w(\alpha) \tag{2.22}$$

which by (2.21) reduces to

$$\langle l_0\rangle = \frac{\langle 0|l_0 \mathrm{e}^{2\Sigma\alpha l_0}|0\rangle}{\langle 0|\mathrm{e}^{2\Sigma\alpha l_0}|0\rangle}. \tag{2.23}$$

Now by definition, the mean mid-point height $\langle l_0 \rangle$ is the first moment of the probability distribution $P_F(l_0)$ for the mid-point height, so we can immediately identify

$$P_F(l_0) = \frac{|\psi_0(l_0)|^2 e^{2\Sigma\alpha l_0}}{\langle 0|e^{2\Sigma\alpha l_0}|0\rangle} \tag{2.24}$$

where the subscript on the left-hand side denotes filling. The same result for $P_F(l_0)$ may be derived in a number of ways and may be generalized to the mean height probability distribution for any distance, x, away from the wedge centre [1, 13].

Analysis of equations (2.21), (2.22), (2.23) and (2.24) allows a complete classification of the critical exponents for filling for $d = 2$ without bulk disorder [1]. Firstly it is clear that the thermodynamic prediction for the filling phase boundary (1.1) is precisely obeyed in this model, since from (2.2) and (2.4) the asymptotic decay of the ground-state wavefunction is $\psi_0(l) \sim e^{2\Sigma\theta_\pi l + \mathcal{O}(l^{1-p})}$ for all potentials $W(l)$ that decay as $l \to \infty$. Thus the distribution function $P_F(l_0)$ is only normalizable for temperatures $T < T_F$ with $\theta(T_F) = \alpha$. In the filling fluctuation (FFL) region corresponding to $p > 1$, the scaling form of the distribution function immediately follows as

$$P_F(l_0) = 2\Sigma(\theta_\pi - \alpha)e^{-2\Sigma(\theta_\pi - \alpha)l_0} \qquad \text{FFL} \tag{2.25}$$

and note that $\theta_\pi - \alpha \approx T_F - T$ as $T \to T_F^-$ so $\beta_0 = \nu_\perp = 1$. For $p < 1$ the transition is described by mean-field-like critical exponents as quoted in the introduction. The filling transition with $p = 1$ is marginal and we will return to this later. Thus the scaling form of the distribution function for fluctuation-dominated filling transitions is the same as that encountered in the SFL regime. Thus for systems with purely short-ranged forces we can conclude that

$$P_F(l_0; \theta_\pi) = P_\pi(l; \theta_\pi - \alpha) \tag{2.26}$$

as stated earlier.

Finally we note that from equation (2.13) and equation (2.21) the explicit expression for the wedge free energy is

$$f_w(\alpha) = \ln\left(\frac{\theta_\pi - \alpha}{\theta_\pi}\right) \tag{2.27}$$

valid in the fluctuation-dominated regime $p > 1$ and for $T < T_F$. Thus the wedge free energy becomes singular as $T \to T_F$. For the model with short-ranged forces defined by equation (2.11) this can be written as

$$f_w(\alpha) = \ln\left(\frac{\lambda - \Sigma\alpha}{\lambda}\right) \tag{2.28}$$

which we will return to later.

3. Filling transitions with bulk random-bond disorder

Following the presentation of section 2, we first recall the pertinent details of Kardar's treatment of wetting at a planar wall with bulk disorder. Further details can be found in [7] and [12]. The model considered is an extension of the interfacial Hamiltonian (2.1) which accounts for extra disorder arising from random bonds. We write

$$H[l] = \int_0^X \left\{ \frac{\Sigma}{2}\left(\frac{\partial l}{\partial x}\right)^2 + V_r(x, l(x)) \right\} \, dx \tag{3.1}$$

where the random variable $V_r(x, l(x))$ is taken to be Gaussian and has the following statistical properties:

$$\overline{V(x, l_i[x])} = 0 \tag{3.2}$$

and

$$\overline{V_r(x, l_i[x])V_r(x', l_i'[x])} - \overline{V_r(x, l_i[x])}][\overline{V_r(x', l_j'[x])} = \Delta\delta(x - x')\,\delta(l_i(x) - l_j'(x')) \tag{3.3}$$

where the overbar denotes a quenched average with respect to bulk disorder. Again setting $k_bT \equiv 1$ we define $\kappa \equiv \frac{1}{2}\,\Delta\Sigma$ which may be interpreted as the inverse length scale induced by the bulk disorder. Thus in the limit $\kappa \to 0$ we should recover the results for the corresponding ordered system. In particular for our wedge problem we will require that as $\kappa \to 0$ we recover the properties of the filling transition presented in section 2. Continuing our discussion for the wetting case, we note that the starting point for the calculations is the replica trick identification

$$\overline{\ln Z_\pi} = \lim_{n\to 0} \frac{\overline{Z_\pi^n} - 1}{n} \tag{3.4}$$

where Z_π^n may be interpreted as the partition function for n non-interacting interfaces in a bulk random medium. Performing the averaging over the disorder variables one finds

$$\overline{Z_\pi^n} = \int \mathcal{D}l_1\ \cdots\ \mathcal{D}l_n\ \mathrm{e}^{-H^{(n)}[l_1,\ldots,l_n]} \tag{3.5}$$

where the multi-field interacting interfacial Hamiltonian is specified by

$$H^{(n)}[l_1, \ldots, l_n] = \int_0^X \left\{ \sum_{i=1}^n \left[\frac{\Sigma}{2}\left(\frac{\partial l_i}{\partial x}\right)^2 + \overline{V}(x, l_i) + U(l_i) - \frac{1}{2}\beta\,\Delta\delta(0) \right] - \Delta \sum_{i<j}^n \delta(l_i - l_j) \right\} \mathrm{d}x. \tag{3.6}$$

From here on, the problem is reduced to a standard transfer-matrix calculation and we can identify

$$\overline{Z_\pi^n}(\{l_i\}; X) = \sum_{m=0}^\infty \Psi_m^{(n)*}(\{l_i\})\Psi_m^{(n)}(\{l_i'\})\mathrm{e}^{-E_m X} \tag{3.7}$$

where $\Psi_m^{(n)}$ is the 'mth'-state wavefunction with eigenvalue E_m for the 'n'-field interacting model and $\{l_i\}$, $\{l_i'\}$ denote the two sets of end-point values for the n interfaces. The spectrum is determined by the solution of the eigenvalue problem

$$\vec{H}^{(n)}\Psi_m^{(n)} = E_m^{(n)}\Psi_m^{(n)} \tag{3.8}$$

where the Hamiltonian operator is specified as

$$\vec{H}^{(n)}[l_1, \ldots, l_n] = n\left(\overline{V} - \frac{1}{2}\beta\,\Delta\delta(0)\right) - \frac{1}{2\Sigma}\sum_{i=1}^n \frac{\partial^2}{\partial l_i^2} - \Delta\sum_{i<j}^n \delta(l_i - l_j) + \sum_{i=1}^n U(l_i). \tag{3.9}$$

For systems with short-ranged forces the fluid potential term in (3.9) may be dropped and replaced by the boundary condition [7]

$$\left.\frac{\partial \ln \Psi_m^{(n)}}{\partial l}\right|_{l=0} = -\lambda \tag{3.10}$$

where we recall from (2.11) that λ is an inverse length scale that may be regarded as a linear measure of the divergence from the wetting temperature of the ordered system. For this choice of model system, the ground state of the eigenvalue problem (3.8) is solved by a Bethe-*ansatz* wavefunction of the form

$$\Psi_0^{(n)} = C_n(\lambda, \kappa) e^{-\lambda \sum l_i + \kappa \sum_{i<j} |l_i - l_j|} \tag{3.11}$$

where C_n is a normalization constant which be evaluated [7, 12] as

$$C_n(\lambda, \kappa) = 2^{n/2} \kappa^{n/2} \left(\frac{\Gamma(\lambda/\kappa + 2n - 1)}{\Gamma(\lambda/\kappa + n - 1)} \right)^{1/2} \tag{3.12}$$

and $\Gamma(x)$ denotes the usual gamma function. The energy of the ground state follows

$$E_0^{(n)} = -\frac{1}{2\Sigma} \left(\lambda^2 + 2\lambda\kappa(n-1) + \frac{2\kappa^2(n-1)(2n-1)}{3} \right) \tag{3.13}$$

from which we can identify the singular part of the free energy via

$$f_{\text{sing}} = -\frac{(\lambda - \kappa)^2}{2\Sigma} \tag{3.14}$$

and hence the contact angle $\theta_\pi = (\lambda - \kappa)/\Sigma$. From this we can conclude that the wetting transition temperature is lowered by the effect of bulk disorder. In addition we note that the critical exponent α_s is equal to zero—just as in the SFL regime for the case of ordered bulk systems. The mean height of the interface can also be calculated in a straightforward manner. From the replica trick we observe that

$$l_\pi = \lim_{n \to 0} \left\{ \frac{1}{n} \sum l_i \right\} \tag{3.15}$$

which by (3.12) reduces to

$$l_\pi = \lim_{n \to 0} \frac{1}{n 2^n} \frac{\partial}{\partial \lambda} \ln C_n \tag{3.16}$$

and hence

$$l_\pi = \frac{\kappa}{(\lambda - \kappa)^2} \tag{3.17}$$

so the critical exponent $\beta_s = 2$. Forgacs *et al* [7] also describe the calculation of the probability distribution function. Formally this can be written in terms of the replica trick many-body wavefunction as

$$P(l) = \lim_{n \to 0} \frac{1}{n} P^{(n)}(l_0) = \frac{\int_0^\infty \cdots \int_0^\infty dl_0^1 \cdots dl_0^n \, \delta(l - l_1) C_n^2 |\Psi_0^{(n)}(\{l_0^{(i)}\})|^2}{\int_0^\infty \cdots \int_0^\infty dl_0^1 \cdots dl_0^n \, C_n^2 |\Psi_0^{(n)}(\{l_0^{(i)}\})|^2} \tag{3.18}$$

the Laplace transform of which can be evaluated explicitly as

$$\hat{P}(p) = \frac{1}{n} \sum_{i=1}^{n} \prod_{j=i}^{n} \frac{(\sigma + 2n - j)j}{(\sigma + 2n - j)j + p/2\kappa}. \tag{3.19}$$

Upon making certain non-trivial simplifying assumptions, Forgacs *et al* [7] argue that this expression can be reinverted to given an explicit for $P(l)$, but the details of this will not be relevant to our discussion here, as will be shown.

We now turn our attention to the wedge geometry and consider the filling transition with random bonds in the bulk. The geometry is identical to that specified in section 2 for the ordered bulk system: we consider a wall with height given by $z(x) = \alpha|x|$ extending over the range

$[-X/2, X/2]$. Periodic conditions on the edge-point interfacial heights $y(x)$ at $x = \pm X/2$ are also applied. The effective interfacial model is defined as

$$H[y] = \int_{-X/2}^{X/2} \mathrm{d}x \left\{ \frac{\Sigma}{2}\left(\frac{\partial y}{\partial x}\right)^2 + V_r(x, y) + W(y - z) \right\} \tag{3.20}$$

which is the natural extension of wedge model (2.14) to disordered systems. Following our earlier description of ordered filling transitions, the partition function or propagator is conveniently decomposed into contributions from interfacial fluctuations between $-X/2$ and 0 and from 0 to $X/2$ by rewriting the Hamiltonian in terms of the local relative height $l \equiv y - z$:

$$\tilde{H}[l] = -\frac{\Sigma\alpha^2 X}{2} + 2\Sigma\alpha(l_e - l_0) + \int_{-X/2}^{X/2} \mathrm{d}x \left\{ \frac{\Sigma}{2}\left(\frac{\partial l}{\partial x}\right)^2 + V_r(x, l) + W(l) \right\}. \tag{3.21}$$

Taking n replicas and averaging over the disorder we immediately find

$$\overline{Z^n} = \mathrm{e}^{n\Sigma X^2/2} \int \mathrm{d}l_{\mathrm{e}}^{(1)} \cdots \mathrm{d}l_{\mathrm{e}}^{(n)} \int \mathrm{d}l_0^{(1)} \cdots \mathrm{d}l_0^{(n)}\, \mathrm{e}^{2\Sigma\alpha\sum_{i=1}^{n}(l_0^{(i)} - l_{\mathrm{e}}^{(i)})} [\overline{Z}_\pi^{(n)}(\{l_{\mathrm{e}}^{(i)}\}, \{l_0^{(i)}\}; X/2)]^2 \tag{3.22}$$

where $\overline{Z}_\pi^{(n)}$ is the planar propagator for the disordered system. In the $x \to \infty$ limit only the ground-state contribution to $\overline{Z}_\pi$ survives, so we can substitute for the appropriate Bethe-*ansatz* wavefunction (3.11). Crucially the additional exponential terms in (3.22) do not change the form of the integrals and effectively redefine the parameter $\lambda \mapsto \lambda - \Sigma\alpha$. It is this fact that makes the model integrable and means that the values of the integrals can be written in terms of modified normalization constants. We define the wedge free energy in exactly the same way as before by subtracting from $\overline{\ln Z}$ a 'bulk' term $\overline{\ln Z'}$ representing the free energy of a tilted interface with wall function $z = \alpha x$ over the range $[-X/2, X/2]$. Again the contributions from the wedge and inverted wedge must be taken into account to give

$$f_w(\alpha) + f_w(-\alpha) = -\lim_{X\to\infty} \ln\left(\frac{\overline{Z}}{\overline{Z'}}\right). \tag{3.23}$$

In terms of the replicated partition functions this is

$$f_w(\alpha) + f_w(-\alpha) = \lim_{n\to 0} \frac{1}{n}(\overline{Z^n} - \overline{Z'^n}) \qquad X \to \infty \tag{3.24}$$

which from (3.22) allows the identification

$$f_w(\alpha) = \lim_{n\to 0} \frac{1}{n}\left(\frac{C_n(\lambda - \Sigma\alpha, \kappa)^2}{C_n(\lambda, \kappa)^2} - 1\right). \tag{3.25}$$

Using the known function form of C_n from the planar distribution (3.12) we conclude that

$$f_w(\alpha) = \Phi(\lambda - \Sigma\alpha) - \Phi(\lambda) \tag{3.26}$$

where $\Phi(x) \equiv \partial \ln \Gamma(x)/\partial x$ is the digamma function. Equation (3.26) is the first important new result of this section and represents an explicit and rather elegant expression for the wedge free energy of a disordered two-dimensional system. Notice that when $\alpha \to 0$ the wedge free energy vanishes, so we recover the planar interfacial free energy. Less trivial is the limit $\kappa \to 0$ corresponding to zero disorder. Here we require that the expression for $f_w(\alpha)$ reduces to the known result derived earlier (2.21) for systems with short-ranged forces in an ordered bulk medium. As $\kappa \to 0$ the argument of each digamma function diverges and we can use Stirling's formula

$$\Gamma(z) \sim z^{z-1/2}\mathrm{e}^{-z} \qquad z \to \infty \tag{3.27}$$

implying that

$$\Phi(z) \sim \ln z - 1 + \frac{1}{2z} \qquad z \to \infty. \tag{3.28}$$

Thus in the limit $\kappa = 0$ we obtain

$$f_w(\alpha) = \ln\left(\frac{\lambda - \Sigma\alpha}{\lambda}\right) \qquad \kappa = 0 \tag{3.29}$$

which is identical to our earlier result for filling with short-ranged forces in the absence of bulk disorder, equation (2.28). This is an important check on the self-consistency of the replica trick calculation.

From (3.28) and (3.33) we can see that the location of the filling transition in the disordered system is given by the condition

$$\alpha = \frac{\lambda - \kappa}{\Sigma} \tag{3.30}$$

which, by virtue of (3.14), is equivalent to

$$\theta_\pi = \alpha \tag{3.31}$$

in precise accord with the thermodynamic prediction. As $T \to T_F^-$, the wedge free energy becomes singular and, from the asymptotics of $\Phi(z)$ as $z \to 0$, we conclude that

$$f_w(\alpha) \sim \left(\frac{\lambda - \Sigma\alpha - \kappa}{\kappa}\right)^{-1}. \tag{3.32}$$

Differentiating this with respect to α yields an expression for the mid-point height in the wedge:

$$l_0 \sim \frac{1}{\kappa(\lambda - \Sigma\alpha - \kappa)^2} \tag{3.33}$$

which implies that the filling critical exponent $\beta_0 = 2$ for this system. Thus the divergence of the mid-point height at a filling transition is characterized by the same critical exponents as those that determine the interface thickness at a planar wetting transition. This equivalence can be made more precise by computing the mid-point height probability distribution $P_F(l_0)$ which from the transfer-matrix theory is given by

$$\begin{aligned} P_F(l_0) &= \lim_{n\to 0} \frac{1}{n} P_F^{(n)}(l_0) \\ &= \frac{\int_0^\infty \cdots \int_0^\infty \mathrm{d}l_0^1 \cdots \mathrm{d}l_0^n \, \delta(l - l_1) \mathrm{e}^{2\Sigma\alpha \sum_{i=1}^n l_0^{(i)}} C_n^2 |\Psi_0^{(n)}(\{l_0^{(I)}\})|^2}{\int_0^\infty \cdots \int_0^\infty \mathrm{d}l_0^1 \cdots \mathrm{d}l_0^n \, \mathrm{e}^{2\Sigma\alpha \sum_{i=1}^n l_0^{(i)}} C_n^2 |\Psi_0^{(n)}(\{l_0^{(i)}\})|^2}. \end{aligned} \tag{3.34}$$

If we now substitute for the Bethe-*ansatz* wavefunctions in this expression, (3.34), it becomes immediately clear that this function can be brought into a precise equivalence with that obtained in the planar case by the simple redefinition $\lambda \mapsto \lambda - \Sigma\alpha$. We then readily conclude that

$$P_F(l_0) = P_\pi(l; \theta_\pi - \alpha) \tag{3.35}$$

since replacing θ_π by $\theta_\pi - \alpha$ in (3.18) is equivalent to this redefinition of λ.

4. Wetting and filling in ordered systems (II)—marginal forces

The explicit calculations of the previous two sections have shown that for systems with short-ranged forces the mid-point height probability distribution function $P_F(l_0)$, near filling, has the same functional form as the height distribution function $P_\pi(l, \theta_\pi)$ at wetting occurring in systems with short-ranged forces. This is true for both ordered and disordered systems. One common feature of two-dimensional wetting in ordered and disordered systems with short-ranged forces is that the value of the specific heat critical exponent $\alpha_s = 0$. Given that this exponent also dictates the singular behaviour of the contact angle $\theta_\pi \sim (T_w - T)^{(2-\alpha_s)/2}$, it is natural to enquire whether or not this vanishing of the critical exponent pertaining to the wetting specific heat is central to the covariance of the probability distribution. To test this hypothesis we return to the case of filling in ordered systems and concentrate on the marginal case $p = 1$. Recall that for $p > 1$ the scaling of $P_F(l_0)$ is the same as that for short-ranged forces. For $p < 1$ the distribution function $P_F(l_0)$ does not show scaling with a single length scale since the transition is mean-field-like and $l_0 \gg \xi_\perp$. The question that we address here is: what is the behaviour of $P_F(l_0)$ at the fluctuation-dominated/MF borderline and how is this related to a wetting distribution function? To this end, consider a binding potential of the form

$$W(l) = -\frac{a}{l} + \frac{w}{l^2} \tag{4.1}$$

where we have included a marginal repulsive interaction. For the planar geometry this potential describes a wetting transition (occurring as $a \to 0^+$) belonging to the WFL/MF borderline. The values of the critical exponents are $\alpha_s = 0$, $\beta_s = 1$, $\nu_\perp = 1$ and $\nu_\parallel = 2$. This model therefore corresponds to a third example of a fluctuation-dominated two-dimensional wetting transition in which the critical exponent $\alpha_s = 0$. The form of the wetting probability distribution function is easily calculated from the solution of the Schrödinger equation (2.4) for the ground-state wavefunction. We find

$$P_\pi(l; \theta_\pi) = N l^{1+\sqrt{1+8w\Sigma}} \mathrm{e}^{-2\Sigma\theta_\pi l} \qquad \text{WFL/MF} \tag{4.2}$$

with $\theta_\pi \propto (T_w - T)$ and with a normalization constant $N = N(w\Sigma, \Sigma\theta_\pi)$. Although the numerical values of the critical exponents at the wetting transition are identical to those pertinent to the SFL regime, the distribution functions are not the same. Indeed the non-trivial short-distance expansion of (4.2) means that fluctuations that bring the interface near to the wall are less likely than in the SFL regime. If we now consider a filling transition occurring in a wedge geometry with the binding potential (4.1), we immediately conclude from the general transfer-matrix result (2.19) that the mid-point height distribution is

$$P_F(l_0) = N l_0^{1+\sqrt{1+8w\Sigma}} \mathrm{e}^{-2\Sigma(\theta_\pi - \alpha) l_0} \qquad \text{WFL/MF} \tag{4.3}$$

which is identical in form to (4.2), with θ_π simply replaced by $\theta_\pi - \alpha$.

5. Discussion

Gathering together all of our results, we are led to the following conclusion: if a two-dimensional planar system exhibits a wetting transition with large fluctuation effects (i.e. with $l_\pi \sim \xi_\perp$) and has a vanishing specific heat exponent $\alpha_s = 0$, then the distribution function $P_\pi(l, \theta_\pi)$ also determines the scaling of the mid-point height probability $P_F(l_0)$ occurring for the filling transition in the non-planar wedge geometry:

$$P_F(l_0) = P_\pi(l, \theta_\pi - \alpha). \tag{5.1}$$

This simple relation completely classifies the allowed critical singularities for fluctuation-dominated filling transitions in two-dimensional systems. In particular, for ordered and disordered systems with short-ranged forces the values of the filling critical exponents are $\beta_0 = 1$ and $\beta_0 = 2$ respectively.

At this stage two pertinent remarks are in order which serve to reinforce the differences between wetting and filling. Firstly, any direct relationship between wetting and filling is a special property of two-dimensional systems. As discussed in the introduction, in higher dimensions, the dominant fluctuations at filling transitions arise due to the anisotropy of the correlation length measured across (ξ_x) and along (ξ_y) the system. Consequently it is important that wetting and filling be considered as quite distinct examples of interfacial critical phenomena with very different critical singularities. In two dimensions, ξ_y is no longer defined and the essential anisotropy does not manifest itself.

Secondly, even for $d = 2$, the full phenomenological descriptions of wetting and filling are clearly not identical. It is not the case that two-dimensional filling may be viewed as a wetting transition shifted to a lower transition temperature. To see this, recall that at a two-dimensional wetting transition the transverse correlation length $\xi_\parallel$ (measuring fluctuations along the substrate) diverges with critical exponent $\nu_\parallel = 2$ as $T \to T_w^-$, whilst at filling the correlation length ξ_x diverges with exponent $\nu_x = 1$ as $T \to T_F^-$. Thus the anisotropy of fluctuations in the vertical (l-) and transverse (x-) directions are quite different for two-dimensional wetting and filling.

Our work is strongly suggestive that there exists a mapping or covariance for the one-point interfacial height probability distribution function for two-dimensional systems (with short-ranged forces) occurring near related phase transitions in different geometries. Whilst we do not know the most general form of this invariance relating the distribution functions for more general types of bounded two-dimensional geometries, it is very tempting to generalize our central result (1.6) and make a conjecture for finite-size effects occurring at filling transitions. This may be tested in further numerically based transfer-matrix and computer simulation studies of various model systems.

For concreteness we deal with Ising spin systems and concentrate on pure thermal excitations where much is already known about finite-size effects at wetting. Consider an Ising strip of width L (which is much greater than the correlation length ξ_b) and of infinite extent in the other (x-) directions. Surface fields H_1 and H_2 act on opposite surfaces of this strip and induce an interface that is free to wander within the confined geometry. For this system a number of detailed predictions have been made (and confirmed) for the form of the one-point probability distribution $P(l; L, T)$ [14, 15]. This is known to exhibit scaling behaviour both at (and above) the wetting temperature T_w of the semi-infinite system. In particular, exactly at T_w, the interface wanders freely between the confining walls and has equal probability of being found at any particular height. Thus

$$P(l; L, T_w) = \frac{1}{L} \tag{5.2}$$

which in turn means that the mean local magnetization $m(z)$ at distance z across the strip is given by

$$m(z) = m_0\left(1 - \frac{2z}{L}\right) \tag{5.3}$$

where m_0 denotes the bulk spontaneous magnetization. These predictions, based on the analysis of interfacial Hamiltonian models [14, 15] and general short-distance expansion expectations, are fully confirmed by Ising model studies [14, 16, 17].

Now consider a finite-size, or double-wedge, system of (vertical) diagonal width L with local fields H_1 and H_2 acting on the respective lower and upper surface layers. One example of this is the finite-size, square Ising model shown in figure 2. This choice of boundary condition induces an interface that stretches across the horizontal diagonal of the square lattice. In the limit $L \to \infty$ the system decouples into two independent wedge geometries each of which exhibits a filling transition at temperature $T_F < T_w$. The lower wedge is filled by up spins whilst the upper wedge is filled by down spins. We now ask what the probability distribution

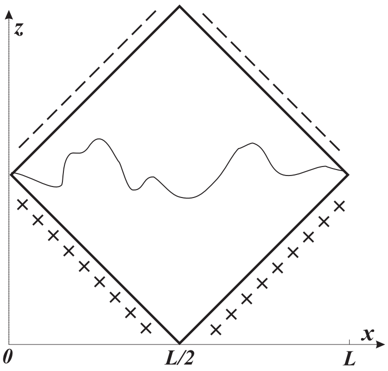

Figure 2. The finite-size double-wedge system. The sides marked with positive signs indicate walls with positive surface field which shows a preference for the up-spin phase and vice versa.

$P_F(l_0; L, T_F)$ is for finding the mid-point interfacial height l_0, for a finite-size system exactly at the (semi-infinite) filling transition temperature T_F. On the basis of the known invariance of $P_F(l_0)$ for the semi-infinite system, we conjecture that the distribution function for the finite-size system is simply related to that of the corresponding planar Ising strip. That is,

$$P_F(l_0; L, T_F) = P(l; L, T_w) \tag{5.4}$$

which implies that the inhomogeneous magnetization profile $m(z, x)$ satisfies

$$m\left(z, \frac{L}{2}\right) = m_0\left(1 - \frac{2z}{L}\right). \tag{5.5}$$

Note that this prediction implies that the vertical, r.m.s. fluctuations of the interface in the z-direction are of the same order as the overall length of the interface in the x-direction. This is quite different to finite-size effects occurring at wetting, where the vertical fluctuations are of order the square root of the interface length (for $d = 2$). Again this serves to emphasize that any invariance of the one-point probability distribution function for wetting and filling does not imply that the two phase transitions have identical character. We also remark that if the conjectured relations (5.4) and (5.5) are confirmed by detailed numerical studies, then the characteristic finite-size scaling properties of $P_F(l_0; L, T_F)$ could provide an effective method for determining the location of the filling transition temperature T_F. For $T > T_F$ we do not anticipate $P_F(l_0; L, T)$ to have a simple scaling form, since this function is characteristic of finite-size effects at complete filling which do not exhibit strong interfacial-fluctuation-related behaviour.

In summary, we have shown by explicit calculation that the one-point probability distribution functions for filling and wetting in (1 + 1)-dimensional ordered and disordered systems are related and have identified the critical exponents that characterize the filling transition in 1+1 dimensions. The calculation for random-bond systems is new and generalizes Kardar's Bethe-*ansatz* techniques to wedge geometries. On the basis of these observations we have made conjectures for the probability distribution and magnetization in finite-size wedge systems which can be tested in future studies.

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